ON AN APPROXIMATE METHOD OF SOLUTION OF PROBLEMS IN PLASTICITY

V. I. Odinokov

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An approximate method of solving the variational equation, constructed on the basis of the principle of virtual variation of the deformed state, with a given equation relating the strains (strain rates) is presented. The stress-strain state is then determined from the solution of the above variational equation. The method is demonstrated on an example of the problem of a strip with a rectangular cross section resting on plane-parallel plates.

1. It is required to determine the velocity and stress fields v_i and σ_{ij}, i, j = 1, 2, 3, from the solution of the variational equation, constructed on the basis of the principle of virtual variation of the displacement rates δv_i

$$\int \sigma_{ij}\delta\xi_{ij}dV - \int X_i \delta v_i dS = 0 \tag{1.1}$$

with a given equation of constraint

$$\xi = \xi_{ij}\delta_{ij} = 0 \qquad \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$
$$\xi_{ij} = {}^1/_2 (v_{i,j} + v_{j,i}) \tag{1.2}$$

and the equations of state

$$\sigma_{ij} = \sigma\delta_{ij} + 2\lambda\xi_{ij}$$
$$\lambda = T / H, \quad T = Hg(H), \quad H = (2\xi_{ij}^*\xi_{ij}^*)^{1/2},$$
$$\xi_{ij}^* = \xi_{ij} - {}^1/_3\xi\delta_{ij} \tag{1.3}$$

where σ is the hydrostatic pressure, X_i are the components of the surface traction on the surface S, and $v_{i,j} = \partial v_i/\partial x_j$. In the above formulas summation over repeated indices i, j is assumed.

If Eqs. (1.3) and (1.2) are taken into account and if it is assumed that the boundaries of the deformed region are prescribed, the variational Eq. (1.1) has the form

$$\delta\{\int[\int TdH]dV - \int X_i v_i dS\} = 0 \tag{1.4}$$

The solution of Eq. (1.4) is the minimum of the functional

$$J = \int[\int TdH]dV - \int X_i v_i dS = \min \tag{1.5}$$

with Eq. (1.2) taken into account.

Let us complete the functional (1.5)

$$J^* = J + \int \sigma\xi dV \tag{1.6}$$

where ∂ is the Lagrange multiplier. Physical considerations indicate that ∂ in Eq. (1.6) represents the hydrostatic pressure [1, 2].

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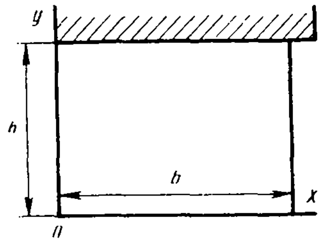

Fig. 1

Let us write the first variation of the functional (1.6)

$$\delta J^* = \int [2\lambda \xi_{ij}{}^* \delta \xi_{ij}{}^* + (\sigma \delta \xi_{ij} + \xi_{ij} \delta \sigma)]\, dV - \int X_i \delta v_i dS = 0 \tag{1.7}$$

We solve the variational Eq. (1.7) by the Ritz method. The approximate solution is sought in the form

$$v_i = v_{i0} + \sum_{m=1} c_{im} v_{im}, \quad i = 1, 2, 3 \tag{1.8}$$

$$\sigma = \sigma_0 + \sum_{k=1} a_k \sigma_k \tag{1.9}$$

Here v_{i0} are functions which satisfy the given conditions on the surface S; σ_0 is a function which satisfies the conditions with respect to σ_{ii} on the surface S; v_{im}, m = 1, (1), n; σ_k, k = 1, (1), and t is a sequence of coordinate functions which satisfy the zero conditions on S; c_{im} and a_k are the Ritz coefficients.

Taking into account Eqs. (1.8) and (1.9), we rewrite Eq. (1.7) in the form

$$\delta J^* = \int \{2\lambda \xi_{ij}{}^* (\xi_{ij}{}^*),_{c_{pm}} \delta c_{pm} + [\sigma (\xi_{ij}),_{c_{pm}} \delta c_{pm}]\, \delta_{ij} + [\xi_{ij} \sigma,_{a_k} \delta a_k]\, \delta_{ij}\}\, dV - \int X_i v_{i,c_{pm}} \delta c_{pm} dS = 0, \quad p = 1, 2, 3 \tag{1.10}$$

In Eq. (1.10) the summation is carried out over the indices i, j; as in Eq. (1.2), the following designations are adopted for the partial derivatives $(\xi_{ij}),_{c_{pm}} = \partial \xi_{ij} / \partial c_{pm}$, etc.

Grouping together the terms which accompany the same variations δc_{pm} or δa_k, we obtain the equations

$$\int [2\lambda \xi_{ij}{}^* (\xi_{ij}{}^*),_{c_{pm}} + \sigma (\xi_{ij}),_{c_{pm}} \delta_{ij}]\, dV - \int X_i v_{i,c_{pm}} dS = 0 \tag{1.11}$$

$$\int \xi_{ij} \sigma,_{a_k} \delta_{ij} dV = 0 \tag{1.12}$$

It follows from Eq. (1.12) that $\xi = 0$.

Let us use $\xi = 0$ to simplify Eq. (1.11) and to obtain a system of equations equivalent to Eq. (1.12). To that end, in the equation $\xi = 0$ we set coefficients which accompany identical functions equal to one another. Thus, we obtain l equations. We have

$$\int [2\lambda \xi_{ij} (\xi_{ij}),_{c_{pm}} + \sigma (\xi_{ij}),_{c_{pm}} \delta_{ij}]\, dV - \int X_i v_{i,c_{pm}} dS = 0 \tag{1.13}$$

$$[b_{1m} c_{1m} + b_{2\alpha} c_{2\alpha} + b_{3\beta} c_{3\beta}] \Big|_{\frac{1}{b_{1m}} v_{1m,1} = \frac{1}{b_{2\alpha}} v_{2\alpha,2} = \frac{1}{b_{3\beta}} v_{3\beta,3}} = 0 \tag{1.14}$$

where $m \in N$, $\alpha \in N$, $\beta \in N$, and N = {0, 1, (1), n}.

The number of parameters t of a_k must equal l (t = l).

Solving systems (1.13) and (1.14) jointly, we find all the independent parameters and, consequently, the velocity and stress fields v_i and σ, σ_{ij}.

Should difficulties in integrating the functional arise, a modified Ritz method [3] can be used; furthermore, values of the independent parameters c_{im} can be found by determining the minimum of the functional (1.5) by means of a numerical method, and then the parameters of a_k can be obtained from the solution of the system (1.13). Since the number of parameters c_{pm} exceeds that of parameters a_k, the possibility arises of selecting among the parameters c_{pm} those for which differentiation would lead to simpler equations.

The stress field can be obtained without recourse to the equations of equilibrium, which follow from Eq. (1.1), and consequently will be satisfied more accurately the more accurately the velocity field is described. The accuracy of computing normal stresses from strain rates with the use of the equilibrium equations depends to a great extent upon the completeness of description of boundary conditions with respect to shear stresses, which is not always a simple task. Consequently, stress fields calculated by means of equilibrium equations very often display drastic qualitative differences, as compared to the true values.

The procedure presented above gives the values of stresses with the same accuracy with which the velocity was determined.

The above method can be used without considerable changes to determine the stress-strain state from the solution of the variational equation constructed on the principle of virtual variation of the state of stress. In that case, the Lagrange multipliers in the equilibrium equations have the physical meaning of displacement rates [2].

2. Let us apply the above method to the problem of settlement of a strip of rectangular cross section resting on plane-parallel plates with rough surfaces.

We assume the strip to be sufficiently long to justify the plane-problem approach.

Bearing the symmetry in mind, we consider one-quarter of the focus of deformation (Fig. 1).

We assume that the contact surface is acted upon by a constant friction stress τ, which, according to Fig. 1, equals

$$\tau = -\psi\tau_S \tag{2.1}$$

where τ_S is the yield point in shear, and ψ is the friction coefficient.

The problem is solved in terms of velocities, considering the moment of deformation. The material being deformed is assumed to be incompressible.

For the case under consideration the variational Eq. (1.1) holds true, as well as the equation of constraint (1.2).

Let us assume that the material being deformed has the properties of a linearly viscous medium

$$T = \mu H$$

The solution of the problem requires satisfaction of the boundary conditions

$$v_x\Big|_{x=0} = 0, \quad v_y\Big|_{y=0} = 0, \quad v_y\Big|_{y=h} = -v_u, \quad \sigma_x\Big|_{x=b} = 0 \tag{2.2}$$

where v_u is the velocity of the instrument displacement, and of the condition of symmetry of flow.

To simplify the solution we shall not fulfill certain boundary conditions with respect to σ_{xy}. Let us write the functions suitable for the velocities v_x and v_y in the form [4]

$$\begin{aligned} v_x &= c_0\frac{x}{b} + c_1\frac{x}{b}\left(1 - 3\frac{y^2}{h^2}\right)\left(1 - \frac{1}{3}\frac{x^2}{b^2}\right), \\ v_y &= -\left[v_u\frac{y}{h} + c_2\frac{y}{h}\left(1 - \frac{y^2}{h^2}\right)\left(1 - \frac{x^2}{b^2}\right)\right] \end{aligned} \tag{2.3}$$

From Eq. (2.3) we find the strain rates

$$\begin{aligned} \xi_x &= c_0\frac{1}{b} + c_1\frac{1}{b}\left(1 - 3\frac{y^2}{h^2}\right)\left(1 - \frac{x^2}{b^2}\right), \\ \xi_y &= -\left[\frac{v_u}{h} + \frac{c_2}{h}\left(1 - 3\frac{y^2}{h^2}\right)\left(1 - \frac{x^2}{b^2}\right)\right] \\ \xi_{xy} &= -\frac{6c_1xy}{bh^2}\left(1 - \frac{1}{3}\frac{x^2}{b^2}\right) + \frac{2c_2xy}{hb^2}\left(1 - \frac{y^2}{h^2}\right) \end{aligned} \tag{2.4}$$

It follows from the condition $\xi = 0$ that the system (1.4) must consist of two equations. The series for σ must contain two unknown parameters a_k, k = 1, 2

$$\sigma = \sigma_0 + a_1(1 - x^2/b^2) + a_2(1 - y^2/h^2)(1 - x^2/b^2) \tag{2.5}$$

The parameter σ_0 is found from the condition

$$\sigma_0 = -2\mu b^{-1}c_0 \tag{2.6}$$

Substituting Eqs. (2.4) and (2.5) into Eq. (1.13) and bearing in mind that $\lambda = \mu$, after differentiation with respect to parameters c_0, c_1, and c_2 and integration, we obtain the system of linear equations

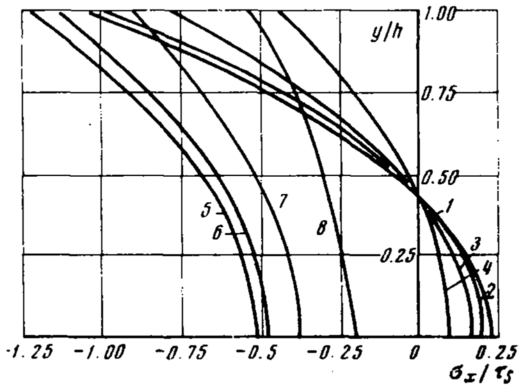

Fig. 2

$$2\mu c_0 h / b + \sigma_0 h + 0.666 h a_1 + 0.444 h a_2 + 0.5\psi\tau_S b = 0$$
$$2\mu\,(0.426 h b^{-1} c_1 + 1.29 b h^{-1} c_1 - 0.212 c_2) + 0.142 h a_2 - 0.833 b\psi\tau_S = 0 \qquad (2.7)$$
$$2\mu\,(0.426 b h^{-1} c_2 + 0.0507 h b^{-1} c_2 - 0.212 c_1) - 0.142 b a_2 = 0$$

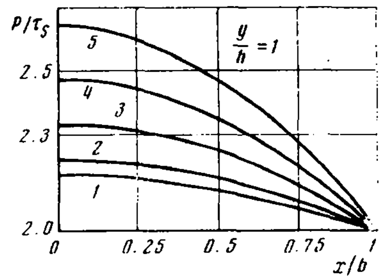

Fig. 3

The system of Eqs. (1.14) has the form

$$c_0 1 / b - v_u / h = 0, \quad c_1 / b - c_2 / h = 0 \qquad (2.8)$$

Solving the system (2.5)-(2.8) jointly, we obtain

$$c_0 = v_u b / h$$
$$c_1 = v_u 0.416\psi \frac{b}{h}\left[\frac{2\mu v_u}{h\tau_S}\left(0.213\frac{h}{b} + 0.648\frac{b}{h} + 0.0253\frac{h^2}{b^2}\right)\right]^{-1}$$
$$c_2 = c_1 h/b$$
$$a_1 = \tau_S\left[\frac{2\mu c_1}{h\tau_S}\left(\frac{h}{b} + 6.04\frac{b}{h}\right) - 4.65\psi\frac{b}{h}\right],$$
$$a_2 = \tau_S\left[5.86\psi\frac{b}{h} - \frac{2\mu c_1}{\tau_S h}\left(1.51\frac{h}{b} + 9.08\frac{b}{h}\right)\right]$$

Figures 2 and 3 show some diagrams of distribution of stresses and specific pressures P in the focus of deformation for $\psi = 0.5$, $2\mu v_u/h\tau_S = 1$.

Thus, Fig. 2 gives the diagrams of distribution of σ_x/τ_S along the height of the focus of deformation in the cross sections x/b = 0, 0.25, 0.5, and 0.75. Numbers 1, 2, 3, and 4 designate the curves which correspond to the cross sections x/b listed above for b/h = 0.5. Numbers 5, 6, 7, and 8 designate the curves which correspond to the same cross sections x/b for b/h = 2. Figure 3 gives the diagrams of specific pressures P/τ_S on the contact surface between the metal and the instrument. Numbers 1, 2, 3, 4, and 5 designate the curves of specific pressure for the corresponding criteria of b/h (b/h = 1.0, 1.5, 2.0, 2.5, and 3.0).

The above diagrams show qualitative agreement with existing solutions and experimental data on the settlement of incompressible materials, despite the nonfulfillment of certain boundary conditions on σ_{xy}.

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